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On the structure of DeWitt supermanifolds

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Abstract

We show that for a wide and most natural class of (possibly infinite-dimensional) Grassmannian algebras of coefficients, the structure sheaf of every smooth DeWitt supermanifold is acyclic (i.e. its cohomology vanishes in positive degree). This result was previously known for finite-dimensional ground algebras and is new even for the original DeWitt algebra of supernumbers Λ_∞. From here we deduce that (equivalence classes of) smooth DeWitt supermanifolds over a fixed ground algebra and of graded smooth manifolds are in a natural bijection with each other. However, contrary to what was stated previously by some authors, this correspondence fails to be functorial; so it happens, for instance, for Rogers' ground algebra B_∞. Finally, we observe that every DeWitt super Lie group is a deformation of a graded Lie group over the spectrum Spec Λ of the ground algebra. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

Supermanifold theory forms an interesting and fruitful chapter in the modern history of the relationship between geometry and physics, one characterized by a diversity of approaches to the core concepts of the theory and even by a certain amount of controversy.

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This chapter hardly deserves to be closed without having properly examined and understood the precise relationship among the existing approaches to the concept of a supermanifold, and moreover unifying them on a solid basis. While the algebro-geometric approach to the concept of a supermanifold, combining sheaf-theoretic (locally ringed superspaces) and functorial (functor of points) methods is undoubtedly most profound and fundamental, the more intuitive models of Rogers and DeWitt cannot be merely thrown away and have to be merged into the mainstream picture. To mention just one example, the existence of infinite-dimensional Lie superalgebras that come from Rogers supergroups but not from supergroups in a more restrictive sense [43] backs our viewpoint.

For a more detailed discussion of the various approaches to supermanifolds we refer the reader to the paper [9].

The present paper aims to contribute to a deeper understanding of the exact relationship between Berezin–Leĭtes–Kostant supermanifolds (graded manifolds) and DeWitt supermanifolds over various choices of ground algebras of 'supernumbers'. Our main results demonstrate that isomorphism classes of DeWitt supermanifolds and those of graded manifolds are in a one-to-one correspondence in the sense that to a given DeWitt supermanifold M with structure sheaf $\mathcal{G}$ one can associate a graded manifold $(M_{\mathrm{B}}, \mathcal{G}_0)$ (where M_{B} is the body of M), and vice versa, given a graded manifold $(X, \mathcal{A})$, one can construct a DeWitt supermanifold $(M, \mathcal{G})$ with $(M_{\mathrm{B}}, \mathcal{G}_0) \cong (X, \mathcal{A})$. The correspondence $(M, \mathcal{G}) \mapsto (M_{\mathrm{B}}, \mathcal{G}_0)$ is always functorial, while the inverse correspondence is only so under special restrictions on the class of ground algebras, excluding, for example, Rogers's ground algebra B_∞. (This fact was overlooked by Leĭtes in [35].) We draw some conclusions for supergroups and state a conjecture: can every supergroup (graded Lie group) be deformed to an 'unconventional' DeWitt super Lie group?

In the beginning of the article we survey the main properties of topological graded-commutative algebras serving as 'algebras of supernumbers' (Section 2). Here we single out the class of complete locally multiplicatively convex graded-commutative algebras of Grassmann origin (as we call them, AMGO algebras) as the most natural class allowing for a substantial theory to be developed and at the same time general enough to include all the previously known examples of such algebras. Then we proceed to the important concept of a G^∞ superfunction and Z-expansion and their relationship with the classical notion of Pringsheim regularity (Section 3). Some of our results are new, e.g. the description of those 'algebras of supernumbers' for which the Z-expansion converges for every smooth function: we isolate a rather simple and easily verified necessary and sufficient condition for an AMGO algebra to admit this property. In particular, the latter is shared by the DeWitt supernumber algebra Λ_∞. In Section 4 we survey an axiomatic approach to supermanifolds in the form previously developed elsewhere by a group of researchers including (at different stages) both of the present authors, and further readjust it to our, somewhat more general, setting. Section 5, which is central to this paper, states the acyclicity of the structure sheaf for smooth DeWitt supermanifolds over an arbitrary AMGO algebra; this result is deduced from the existence of supersmooth partitions of unity on an arbitrary DeWitt supermanifold, which result is also new. The correspondence between DeWitt supermanifolds and graded manifolds is discussed in Section 6. Finally, Section 7 contains applications to supergroups.

2. Ground algebras

The letter $\mathbb{K}$ will always denote a basic field, i.e., a fixed field with valuation, complete with respect to the associated metric. While the two cases of importance are those where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, much of the theory makes sense in a more general setting. Let a unital graded-commutative $\mathbb{K}$-algebra Λ be equipped with a Hausdorff topology, $\mathfrak{T}$. (By 'graded' we always mean '$\mathbb{Z}_2$-graded'.) We refer to the pair $(\Lambda, \mathfrak{T})$ as a *topological graded-commutative algebra* if it forms a topological vector space, the multiplication is continuous, and both the odd and the even sector, Λ_i, $i = 0, 1$, are closed. If $\mathfrak{T}$ is locally convex, then we call $\Lambda = (\Lambda, \mathfrak{T})$ a *locally convex graded-commutative algebra.* We say that Λ is an *Arens–Michael graded-commutative algebra* if in addition Λ is an Arens–Michael topological algebra in the sense of [30], i.e., Λ is complete (as a locally convex space) and *locally m-convex* [1,38]. The latter means that the topology is determined by the collection of all continuous submultiplicative prenorms. (A prenorm p on an algebra A is called *submultiplicative* if $p(xy) \leq p(x)p(y)$ for all $x, y \in A$.) Equivalently, a topological algebra is locally m-convex if a neighbourhood basis at zero is formed by convex circled open multiplicative semigroups (*discs*). Consequently, Arens–Michael algebras are exactly those topological algebras represented as projective limits of Banach algebras.

We say that an Arens–Michael graded-commutative algebra Λ is an *Arens–Michael algebra of Grassmann origin,* or simply an *AMGO algebra,* if it is topologically generated by the set $\Lambda_1 \cup \{1\}$, i.e., the values of $\mathbb{K}$-polynomial functions at odd elements are everywhere dense in Λ. This definition extends our previous definition of *Banach algebras of Grassmann origin* or BGO algebras, introduced elsewhere [8,9], and is motivated by our wish to see in the class of ground algebras under consideration DeWitt's 'algebra of supernumbers', Λ_∞ [22–24,37,40], which is non-Banach. The following description is obtained by a straightforward application of the known results about Arens–Michael algebras (cf. [1,30,38]).

Proposition 2.1. *AMGO algebras are exactly those unital graded-commutative locally convex algebras obtained as projective limits of families of Banach algebras of Grassmann origin. In other words, a unital graded-commutative locally convex algebra Λ is AMGO if and only if there exists an inverse system $(B_\alpha, \varpi_\alpha^\beta, A)$ of BGO algebras B_α, $\alpha \in A$ and continuous graded unital algebra homomorphisms $\varpi_\alpha^\beta : B_\beta \to B_\alpha$, $\beta, \alpha \in A$, such that $\Lambda = \lim_{\leftarrow \alpha\in A} B_\alpha$.*

The class of AMGO algebras is probably the most natural class of ground algebras for superanalysis allowing for a substantial theory to be developed. All concrete examples of topological graded-commutative algebras known to us and so far used as 'supernumber algebras' are such. The following list is probably not comprehensive.

1. Finite-dimensional Grassmann algebras [12]. More generally, every finite-dimensional local graded-commutative algebra is AMGO; some examples of such algebras other than Grassmann ones can be found in [31,52].

2. Rogers' Banach–Grassmann algebra B_∞, the Banach algebra completion of the Grassmann algebra $\wedge(\infty)$ with countably infinitely many anticommuting generators equipped with the l_1-type norm [29,46].
3. The previous construction is a particular case of the free unital graded-commutative Banach algebra on a normed space, $\widehat{\wedge}(E)$, introduced in [41]. Namely, $B_\infty = \widehat{\wedge}(l_1)$.
4. All the algebras mentioned in cases 2 and 3 form examples of Banach–Grassmann algebras [29], and some of them are Grassmann–Banach algebras [28]. All Banach–Grassmann and all Grassmann–Banach algebras are BGO algebras.
5. DeWitt's ground algebra, Λ_∞ [16,18–20,22–24], is the projective limit $\overset{\lim}{\longleftarrow}_{n\to\infty} \wedge(n)$ of finite-dimensional Grassmann algebras (for an explicit description of its topology see [37]). It is a Fréchet AMGO algebra which is non-Banach, and being historically the first example of an infinite-dimensional ground algebra (and probably the most convenient one), it fully justifies pushing our setting towards its present limits.
6. The infinite-dimensional Grassmann algebra $\wedge(\infty)$, equipped with the finest locally convex topology, is also an AMGO algebra. It was studied in [32,47].
7. The algebras in cases 6 and 7 are particular examples of varietal free unital graded-commutative Arens–Michael algebras [40]. All such algebras have the AMGO property.

Denote by J_Λ the closed ideal of an AMGO algebra Λ generated by the odd sector. It is easily verified to form the (unique) maximal graded ideal in Λ, since it consists of all non-invertible elements. Thus, every AMGO algebra is a local graded-commutative algebra in the sense of [36]. The factor-algebra Λ/J is naturally isomorphic to the basic field $\mathbb{K}$, and the augmentation homomorphism $\beta_\Lambda : \Lambda \to \mathbb{K}$ is usually called the *body map*. The complementary mapping $\sigma_\Lambda = \mathrm{Id}_\Lambda - \beta_\Lambda$ is known as the *soul map*. Each element $x \in \Lambda$ uniquely decomposes into its *body part* and *soul part*, $x = \beta_\Lambda(x) + \sigma_\Lambda(x) \equiv x_B + x_S$, where $x_B \in \mathbb{K}$ and $x_S \in J_\Lambda$.

Definition 2.2. The *annihilator* of a unital graded-commutative algebra Λ is the subset $\Lambda^\perp = \{x \in \Lambda : \forall y \in \Lambda_1,\ xy = 0\}$.

Every finite-dimensional graded-commutative algebra (in particular, every finite-dimensional Grassmann algebra) has a non-trivial annihilator. The algebras B_∞, Λ_∞, $\wedge(\infty)$, as well as free graded-commutative Banach algebras over infinite-dimensional purely odd Banach spaces and free graded-comutative Arens–Michael algebras over infinite-dimensional purely odd locally convex spaces all have trivial annihilators. For more on this, see [31,39–41].

The proof of the following statement is easy and will be omitted. The projective tensor product is that of locally convex spaces [27].

Proposition 2.3. *The completed projective tensor product, $\Lambda\hat{\otimes}_\pi\Gamma$, of two arbitrary AMGO algebras, Λ and Γ, formed over the basic field $\mathbb{K}$, is again an AMGO algebra. The annihilator of the algebra $\Lambda\hat{\otimes}_\pi\Gamma$ is trivial if and only if the annihilators of both Λ and Γ are trivial.*

3. G^∞ functions and Z-expansion

Let Λ be a unital graded-commutative $\mathbb{K}$-algebra. A graded Λ-module M is *free* if it is isomorphic to one of the form $\Lambda \otimes V$, where V is a graded $\mathbb{K}$-vector space. We will be only considering free modules of *finite rank*, i.e., those of the form $\Lambda^{m|n} = \Lambda \otimes \mathbb{K}^{m|n}$, where $m, n \in \mathbb{N}$ and $\mathbb{K}^{m|n}$ is the standard graded vector space of dimension $m|n$. It is not difficult to prove that the rank $m|n$ of a free module is invariant under the choice of its tensor product representation. For more on free modules, see [36, Ch. 3].

The *body map* $\beta^{m|n} : \Lambda^{m|n} \to \mathbb{K}^{m|n}$ is obtained by tensoring β_Λ with $\mathrm{Id}_{\mathbb{K}^{m|n}}$.

A *linear superspace* $\Lambda^{m,n}$ of dimension (m, n) is the even part of a free Λ-module of rank $m|n$. In particular, it carries a natural structure of Λ_0-module. The corresponding body map $\beta^{m,n} : \Lambda^{m,n} \to \mathbb{K}^m$ is the restriction of $\beta^{m|n}$ to $\Lambda^{m,n} = (\Lambda^{m|n})_0$. The product topology on $\Lambda^{m,n} \cong (\Lambda_0)^m \times (\Lambda_1)^n$ is sometimes called the *Rogers*, or *fine*, topology. The *DeWitt* (or *coarse*) *topology* on $\Lambda^{m,n}$ is the weakest topology making $\beta^{m,n}$ continuous, where the topology on $\mathbb{K}^m$ is of course the usual product topology induced by the valuation on $\mathbb{K}$. If one denotes $\widetilde{A} = (\beta^{m,n})^{-1}(A)$ for a subset $A \subseteq \mathbb{K}^m$, then DeWitt open subsets of $\Lambda^{m,n}$ are exactly those subsets of the form $\tilde{A}$ that are open in the fine topology.

A mapping $\Lambda^{m,n} \to \Lambda^{p,q}$ is called *superlinear* if it can be represented in an obvious way by means of a graded $(m+n) \times (p+q)$-matrix with entries from $\Lambda_0 \cup \Lambda_1$. (For more on this, see [22–24,28,31,46,50,52].)

Let $U \subseteq \Lambda^{m,n}$ be DeWitt open. A mapping $f : U \to \Lambda^{p,q}$ is *superdifferentiable* at a point $x \in U$ if it is Gâteaux differentiable as a mapping between locally convex spaces and the differential $D_x f$ is superlinear. In the case of purely even dimension ($n = 0$), the partial derivatives are uniquely defined as the entries of the corresponding matrix representing the differential. This observation enables one to talk of higher derivatives. In a particular case of a mapping $f : U \to \Lambda$, where $U \subseteq \Lambda^{m,0}$ is DeWitt open, one calls f a G^∞*superfunction* if it has supersmooth derivatives of all orders. More generally, this concept of G^∞-mappings makes sense for $\Lambda^{p|q}$-valued mappings.

A great deal of trouble [15,31,46,52] was caused in the past by the fact that this construction in general does not work properly for mappings between superspaces whose dimension is not purely even, unless the ground algebra satisfies some special restrictions; see [5,48,49] for a discussion.

Since every G^∞ superfunction $f : U \to \Lambda$, where $U \subseteq \Lambda^{m,0}$ is DeWitt open, is infinitely smooth as a mapping between (open subsets of) locally convex spaces, the techniques of global analysis become applicable. In particular, the restriction, f_*, of f to $U_* = U \cap \mathbb{K}^m$, which is an open subspace of $\mathbb{K}^m$, is a (Gâteaux) C^∞ mapping $f_* : U_* \to \Lambda^{m,0}$. (If Λ is Banach, then of course every Gâteaux C^∞ mapping is also Fréchet C^∞, and since Arens–Michael algebras are projective limits of Banach algebras, we are always able to reduce the situation to the Fréchet differentiability setting.) In general, this restriction is not a generic C^∞-mapping from a finite-dimensional space to $\Lambda^{p,0}$ treated as a locally convex space, and the restrictions stemming from f being G^∞ can be quite substantial. If we denote by $\mathcal{C}_*(U, \Lambda)$ the algebra of all mappings of the form $f_* \equiv f|_U : U_* \to \Lambda$, where $f \in G^\infty(\tilde{U}, \Lambda)$, then $\mathcal{C}_*(U, \Lambda) \subseteq C^\infty(U, \Lambda)$, but this inclusion is often proper [28,40].

We will now extend several results, previously established for Banach algebras of Grassmann origin [8], to our more general framework of Arens–Michael algebras of Grassmann origin.

Lemma 3.1. *The mapping $f \mapsto f_*$ is injective. In other words, if $f : \tilde{U} \to \Lambda$ is a G^∞ function, then the condition $f|_U \equiv 0$ implies $f \equiv 0$.*

Proof. Let $f : \tilde{U} \to \Lambda$ be an arbitrary G^∞ function. The algebra Λ contains an everywhere dense nilpotent subalgebra, A (a graded unital subalgebra of Λ algebraically generated by Λ_1). Therefore, the Taylor series of f is terminating at every point of the form $x+\alpha$, where $x \in U$, $\alpha \in A$, after k terms, where $k = k(\alpha)$ is the class of nilpotency of α. Since k remains the same for all scalar multiples $t\alpha$, $t \in \mathbb{K}$, the restriction of f to the affine line $x + t\alpha, t \in \mathbb{K}$ is k-polynomial in t, with all the derivatives of the order $> k$ vanishing for all values of t, and in particular the sum of the terminating Taylor series of f at every point of the form $x + \alpha$ equals $f(x + \alpha)$.

Now assume that $f|_U \equiv 0$. By the above, $f(x + \alpha) = 0$ for each $x \in U$ and $\alpha \in A$. But the points of the form $x + \alpha$ are everywhere dense in $\tilde{U}$, whence $f \equiv 0$ as required. □

Corollary 3.2. *The restriction mapping $f \mapsto f_*$ establishes an isomorphism between the graded Λ-algebras $G^\infty(U, \Lambda)$ and $\mathcal{C}_*(U, \Lambda)$.*

Our next goal is twofold: to invert the restriction mapping $f \mapsto f_*$ and to obtain some idea of the 'size' of $\mathcal{C}_*(U, \Lambda)$ as a subalgebra of $\mathcal{C}^\infty(U, \Lambda)$.

The *Z-expansion* of a C^∞-mapping $f : U \to \Lambda$ (also called *Grassmann analytic continuation* or *soul expansion*), where $U \subseteq \mathbb{K}^m$, is a mapping $Z(f) : \tilde{U} \to \Lambda$ (provided it exists), defined by

$$Z(h)(x) = \sum_{j=0}^{\infty} \frac{1}{j!} D^{(j)} h_{\beta^{m,0}(x)}(s^{m,0}(x))$$

for $h \in \mathcal{C}_*(U, \Lambda)$ and all $x \in \tilde{U}$ [12,23,29,46]. Notice that the jth differential $D^{(j)}h_{\sigma^{m,0}(x)}$ of h at the point $\beta^{m,0}(x)$ acts on $\Lambda^{m,0} \times \cdots \times \Lambda^{m,0}$ (j times) by extending by Λ_0-linearity its action on $\mathbb{K}^m \times \cdots \times \mathbb{K}^m$. Alternatively, the Z-expansion can be rewritten as follows:

$$Z(h)(x) = \sum_{|J|=0}^{\infty} \frac{1}{J!} \left(\frac{\partial^{|J|}h}{\partial x^J}\right)_{\sigma^{m,0}(x)} (s^{m,0}(x))^J,$$

where J is a multi-index.

The behaviour of Z-expansion is described by the following two important theorems.

Theorem 3.3. *Let Λ be an AMGO algebra, let m be a positive integer and U be an open subset of $\mathbb{R}^m$. For an arbitrary G^∞ function f on $\tilde{U}$, the Z-expansion of the restriction f_* of f to U converges to f. The convergence is uniform on compacta* *lying in any 'soul fibre'* $\{x\}^{\sim}$ *with* $x \in U$. *For any* $i = 1, \ldots, m$ *the following holds:* $\partial f/\partial x^i = Z(\partial f_*/\partial x^i)$.

Proof. For BGO algebras the result was established in [8]. Now consider the case of a general AMGO algebra, Λ. Let p be an arbitrary submultiplicative continuous prenorm on Λ. Denote by $\widehat{\Lambda}_p$ the BGO algebra associated to p, i.e., the completion of the factor algebra of Λ by the ideal $\{x \in \Lambda : p(x) = 0\}$ normed in an obvious way, and by $\pi_p : \Lambda \to \widehat{\Lambda}_p$ the continuous factor homomorphism. According to the above mentioned result for BGO algebras, the Z-expansion of $\pi_p \circ f|_U$ converges to some G^∞ function $f_p : \widetilde{U} \to \widehat{\Lambda}_p$ uniformly on compacta lying in the soul fibres of $\widetilde{U}$, where $\tilde{U}$ is the cylinder in $(\widehat{\Lambda}_p)^{m,0}$ over U. It is easy to see that the G^∞ functions f_p, as p runs over all submultiplicative continuous prenorms on Λ, commute with the natural continuous graded algebra homomorphisms $\widehat{\Lambda}_p \to \widehat{\Lambda}_q$, whenever $p \geq q$. Consequently, there exists a unique continuous mapping $f_\bullet : \widetilde{U} \to \Lambda$ such that $\pi_p \circ f_\bullet = f_p$ for every p, where $\widetilde{U}$ is formed in the superspace $\Lambda^{m,0} \cong \overset{\lim}{\longleftarrow}{}_p(\widehat{\Lambda}_p)^{m,0}$. Furthermore, $f_\bullet$ is easily checked to be G^∞ and to satisfy $f_\bullet|_U = f|_U$. Lemma 3.1 implies that $f_\bullet = f$. In the same way one obtains the statement about the derivatives. □

Corollary 3.4. *The* Z*-expansion establishes an isomorphism of graded* Λ*-algebras*

$$Z : \mathcal{C}_*(U, \Lambda) \to \mathcal{C}^\infty(\widetilde{U}, \Lambda),$$

which is the two-sided inverse to the restriction mapping $f \mapsto f_*$.

Corollary 3.5. *Let* $x, y \in \tilde{U}$ *belong to the same soul fibre, i.e.,* $\beta(x) = \beta(y)$. *Then for every* G^∞ *function* $f : \widetilde{U} \to \Lambda^{p,q}$ *the elements* $f(x)$ *and* $f(y)$ *belong to the same soul fibre.*

Proof. It is enough to prove the fact for $p = q = 1$, in which case it follows from comparing the values of Z-expansion of $f|_U$ at x and y. □

Recall that a function $f : U \to \mathbb{R}$ is called *Pringsheim regular* [54] if at any point $x \in U$ of its domain of definition the Taylor series of f converges pointwise in a suitably chosen neighbourhood of x, not necessarily to f itself. Of course, every $\mathbb{K}$-analytic function is Pringsheim regular, and in the complex case $\mathbb{K} = \mathbb{C}$ the converse also holds. However, for some basic fields $\mathbb{K}$, including that of the real numbers, the class of Pringsheim regular functions is strictly wider than that of C^ω functions (while still narrower than the class of C^∞ functions): the universally known example of a Pringsheim regular real function that is not real-analytic is given by $f(x) = \exp(-1/x^2)$ for $x \neq 0$ and $f(0) = 0$. From the latter example it also follows that the familiar bell functions, typically used for constructing partitions of unity, are Pringsheim regular (though never real-analytic), and this observation is important for what follows.

For any open $U \subseteq \mathbb{K}^m$ we will denote by $C^P(U, E)$ the graded-commutative algebra of all Pringsheim regular functions on U taking values in a locally convex space E.

Theorem 3.6. *Let Λ be an AMGO-algebra, let m be a positive integer and U be an open subset of K^m. For an arbitrary Pringsheim regular function f on U, the Z-expansion $Z(f)$ converges to a unique G^∞ function extending f over the DeWitt open set $\widetilde{U}$.*

Proof. This result was established in [8] in the case where Λ is a BGO algebra. The general case is obtained from it by applying Lemma 3.1 and an argument used in the proof of Theorem 3.3. □

It is clear that the Z-expansion of each C^∞ function terminates and therefore converges in the case where the ground algebra Λ is finite-dimensional. But in general this is far from being the case. This fact was observed in [28], and later it was shown in [42] that the Z-expansion of a C^∞ function f converges over the Rogers' algebra B_∞ if and only if f is Pringsheim regular, i.e., $G^\infty(U, B_\infty) \cong C^P(U, B_\infty)$.

It is possible to obtain a description of those ground algebras Λ for which the Z-expansion converges for all infinitely smooth functions. The following result seems to be new. Say that a topological algebra A is *weakly pronilpotent* if every neighbourhood of zero, U, contains a two-sided ideal I such that every element of the factor algebra A/I is nilpotent. For a locally m-convex topological algebra A this condition is equivalent to the following: for every $x \in A$ and every continuous multiplicative prenorm p there is a natural n with $p(x^n) = 0$. In particular, a Banach algebra Λ is weakly pronilpotent if and only if every element of Λ is nilpotent.

Theorem 3.7. *Let Λ be an AMGO algebra. The following conditions are equivalent. (i) For every $m \in \mathbb{N}$, every open $U \subseteq \mathbb{R}^m$ and every C^∞ function $f : U \to \Lambda$ the Z-expansion $Z(f)$ converges at all points of $\widetilde{U}$; (ii) Λ has a weakly pronilpotent radical J_Λ.*

Proof. (i) ⇒ (ii): Assuming that (ii) does not hold, there must exist an *even* element $\kappa \in J_\Lambda$ and a submultiplicative continuous prenorm p on Λ with $p(\kappa^n) > 0$ for all n. Using the Fourier transform in a standard way as e.g. in [40], one can construct a C^∞ function f whose Taylor coefficients at some point $x \in U$ equal $c_n = n!p(\kappa^n)^{-1}$, $n \in \mathbb{N}$. Clearly, $Z(f)$ diverges at x.

(ii) ⇒ (i): it is a direct verification. □

As a particularly important corollary, the Z-expansion converges for all C^∞ functions over the ground algebras Λ_∞ and $\wedge(\infty)$.

We conclude this section with the following concept which, though auxiliary in nature, is relatively important. A G^∞-mapping $f : \widetilde{U} \to \Lambda^{p,q}$ is called an H^∞-*mapping* (cf. [5,23]) if $f(U) \subseteq \mathbb{K}^p$. It follows from Theorem 3.3 that all H^∞ functions are obtained via Z-expansion of those C^∞-mappings $f : U \to \mathbb{K}^p$ for which $Z(f)$ converges. Theorem 3.6 tells us that every Pringsheim regular function $f : U \to \mathbb{K}^p$ extends to an H^∞ function $f : \widetilde{U} \to \Lambda^{p,q}$. It follows from Theorem 3.7 that if the ground algebra has weakly pronilpotent radical, then every C^∞-mapping $f : U \to \mathbb{K}^p$ gives rise to an H^∞ function. It is worth stressing that over the ground algebra B_∞ the H^∞ functions are exactly Z-expansions of Pringsheim regular mappings $f : U \to \mathbb{K}^p$.

4. R^∞-supermanifolds

Our approach to supermanifolds with coefficients in an arbitrary AMGO algebra, Λ, is free from certain deficiencies inherent in the approaches by Rothstein [48,49] and enables one to obtain, for different choices of Λ, all the previously known types of supermanifolds, including the Berezin–Leĭtes–Kostant supermanifolds in the case $\Lambda = \mathbb{K}$ and the original DeWitt supermanifolds in the case $\Lambda = \Lambda_\infty$. For a more detailed presentation (though in a somewhat more restrictive setting, that of BGO algebras), cf. [8].

Here we will only be considering the real case, $\mathbb{K} = \mathbb{R}$. Let Λ be a real AMGO algebra. A *superspace* over Λ is a triple $(X, \mathcal{A}, \mathrm{ev})$, where X is a paracompact topological space, $\mathcal{A}$ is a sheaf of graded-commutative Λ-algebras, and $\mathrm{ev} : \mathcal{A} \to \mathcal{C}_X$ a morphism of sheaves of graded Λ-algebras (where $\mathcal{C}_X$ denotes the sheaf of germs of Λ-valued continuous functions on X). We shall sometimes write $\tilde{\varphi}$ for $\mathrm{ev}(\varphi)$. A *morphism of superspaces* $(f, f^\sharp) : (X, \mathcal{A}, \mathrm{ev}^X) \to (Y, \mathcal{B}, \mathrm{ev}^Y)$ is a pair formed by a continuous map $f : X \to Y$ and a morphism of sheaves of graded Λ-algebras $f^\sharp : \mathcal{B} \to f_*\mathcal{A}$ satisfying $\mathrm{ev}^X \circ f^\sharp = f^* \circ \mathrm{ev}^Y$.

The morphism $\mathrm{ev} : \mathcal{A} \to \mathcal{C}_X$ enables one to evaluate germs of *superfunctions* – i.e. sections of $\mathcal{A}$ – at a given point $p \in X$. We define the graded ideal $\mathfrak{L}_p$ of the stalk $\mathcal{A}_p$ formed by the germs of superfunctions vanishing at p

$$\mathfrak{L}_p = \{\varphi \in \mathcal{A}_p | \tilde{\varphi}(p) = 0\}.$$

An R^∞-*supermanifold* of dimension (m, n), where $m, n \in \mathbb{N}$, is a superspace $(X, \mathcal{A}, \mathrm{ev})$ satisfying the following four axioms.

Axiom 1. The graded $\mathcal{A}$-dual of the sheaf of derivations, $\mathcal{D}er^*\mathcal{A}$, is a locally free graded $\mathcal{A}$-module of rank (m, n). Every point $p \in X$ has an open neighbourhood U with sections $x^1, \ldots, x^m \in \mathcal{A}(U)_0$, $y^1, \ldots, y^n \in \mathcal{A}(U)_1$ such that $\{\mathrm{d}x^1, \ldots, \mathrm{d}x^m, \mathrm{d}y^1, \ldots, \mathrm{d}y^n\}$ is a graded basis of $\mathcal{D}er^*\mathcal{A}(U)$ over $\mathcal{A}(U)$.

Axiom 2. Given a coordinate chart $(U, x^1, \ldots, x^m, y^1, \ldots, y^n)$, the assignment

$$p \mapsto (\tilde{x}^1(p), \ldots, \tilde{x}^m(p), \tilde{y}^1(p), \ldots, \tilde{y}^n(p))$$

defines a homeomorphism of U onto an open subset in $\Lambda^{m,n}$.

Axiom 3. For every $p \in X$ the ideal $\mathfrak{L}_p$ is finitely generated.

The locally convex graded Λ-algebra topology in the algebras of sections $\mathcal{A}(U)$, referred to in the last remaining axiom, is that of uniform convergence on compacta with all derivatives. Namely, let $\mathcal{D}(\mathcal{A})$ be the sheaf of differential operators over $\mathcal{A}$, i.e., the graded $\mathcal{A}$-module generated multiplicatively by $\mathcal{D}er\ \mathcal{A}$ over $\mathcal{A}$. The topology is given by the family of prenorms

$$p_{L,K}(\varphi) = \max_{p \in K} p(\widetilde{L(\varphi)}(p))_\Lambda,$$

with $L \in \mathcal{D}(\mathcal{A}_{|U})$, $K \subset U$ compact, and p runs over the collection of all continuous submultiplicative prenorms on Λ.

Axiom 4. For every open subset $U \subset X$, the topological algebra $\mathcal{A}(U)$ is complete Hausdorff.

Example 4.1. The *standard,* or *model, supermanifold,* $(\Lambda^{m,n}, \mathcal{G}_{m,n})$, is obtained as follows. The underlying topological space of this supermanifold is $\Lambda^{m,n}$. To define the structure sheaf $\mathcal{G}_{m,n}$ denote by $\hat{\mathcal{G}}_m^\infty$ the sheaf of germs of G^∞ functions on $\Lambda^{m,0}$, and by $p : \Lambda^{m,n} \to \Lambda^{m,0}$ the coordinate projection. We set

$$\mathcal{G}_{m,n} = p^{-1}\hat{\mathcal{G}}_m^\infty \otimes_{\mathbb{K}} \wedge(n).$$

The evaluation morphism at a point $x = (x_1, \ldots, x_m, \xi_1, \ldots, \xi_n) \in \Lambda^{m,n}$ is given on elementary tensors by $\mathrm{ev}_x(f \otimes \phi) = f(x_1, \ldots, x_m) \cdot \phi(\xi_1, \ldots, \xi_n)$, where $\phi \in \wedge(n)$ is a polynomial in n anticommuting variables so the substitution of $\xi_1, \ldots, \xi_n$ into ϕ makes sense.

By Axiom 3 one can prove – via a graded version of Nakayama's lemma – the existence of local Taylor expansions exactly as in [7,8]. One can further prove that any (m, n) dimensional supermanifold is locally isomorphic to the standard supermanifold $(\Lambda^{m,n}, \mathcal{G}_{m,n})$ as in [7,8]. This implies that locally superfunctions on an R^∞-supermanifold admit the so-called *superfield expansion,* i.e., a superfunction can be represented in every coordinate neighbourhood as a finite polynomial in odd local variables with coefficients being G^∞ functions depending on even local variables.

The fact that we are working with Arens–Michael rather than Banach algebras never leads to extra complications on any scale. Only considerations of space prevent us from reproducing proofs in all the details. Let us just mention that an important technical result of an auxiliary nature on the density of polynomials in the rings of superfunctions on $\Lambda^{m,n}$, which was proved in [8, Appendix], for BGO algebras Λ, is extended to AMGO-algebras Λ in a straightforward fashion by means of the technique used in the proof of Theorem 3.3.

As it was stressed repeatedly in the past [7–9], the concept of an R^∞-supermanifold extends the usual notion of smooth or complex analytic manifold, in which case the ground algebra Λ reduces to the basic field $\mathbb{K}$. When $\Lambda = \mathbb{K}$ but $\mathcal{A}$ is a genuine sheaf of graded algebras, the notion of Berezin–Leĭtes–Kostant supermanifold (graded manifold) [12,13,33–36] is recovered. Finally, in the case of a finite-dimensional Grassmann algebra $\Lambda = \wedge(n)$, the four axioms lead to the definition of G-supermanifold [4,5].

Remark that, exactly as in the case of graded manifolds [34,35], superfunctions on a supermanifold X can be identified with supermanifold morphisms $X \to \Lambda$, where the ground algebra Λ is thought of as the model supermanifold $\Lambda^{1,1}$. Furthermore, the class of all R^∞-supermanifolds over a fixed AMGO algebra, Λ, together with supermanifold morphisms between them can be given the structure of a category (the composition of morphisms is not quite straightforward, cf. the case of graded manifolds in [34,35]), but we do not pursue this topic any further.

Let us also describe how the notion of G^∞ function of both even and odd variables fits within the present setting. Rogers [46] introduced them as mappings $f : \Lambda^{m,n} \to \Lambda$ which are polynomial in the odd variables with coefficients which are G^∞ in the sense previously discussed. Of course such coefficients are in general not uniquely determined. One should notice that whenever one considers an R^∞ supermanifold $(M, \mathcal{A}, \mathrm{ev})$, the sheaf of G^∞ functions on M coincides with the image of the morphism ev. A notion of G^∞ mapping between supermanifolds can be similarly introduced.

It is important to stress the following well-known phenomenon: in general, a superspace morphism $f : M \to N$ between two R^∞-supermanifolds cannot be restored from the totality of its values at the points of M, i.e., f cannot be recovered if we know the underlying G^∞-mapping $\mathrm{ev}(f)$ alone. However, this becomes possible if the ground algebra Λ has trivial annihilator in the sense of Definition 2.2. The following observation is well known and has been previously established for various classes of ground algebras Λ (cf. e.g. [29,31,47,52]).

Proposition 4.2. *Let Λ be an AMGO algebra. The following two conditions are equivalent: (i) every superspace morphism $f : M \to N$ between two arbitrary R^∞-supermanifolds over Λ is restored from its values at the points of M, i.e., the mapping* ev : $R^\infty(M, N) \to G^\infty(M, N)$ *is an injection; (ii) Λ has trivial annihilator.*

We will also need the following new result. Recall that the basic field $\mathbb{K} = \mathbb{R}$.

Lemma 4.3. *Let f be an R^∞ superfunction on an R^∞ supermanifold X over an AMGO algebra Λ having the property: for every $x \in X$, $\beta_\Lambda(f(x)) \neq 0$. Then f has a multiplicative R^∞-inverse, i.e., there exists an R^∞-superfunction g with $fg \equiv 1$.*

Proof. By covering X with open coordinate charts on which the real-valued continuous function $\beta \circ \mathrm{ev}(f)$ is bounded and uniformly separated from zero and subsequently multiplying f by a constant if necessary, we can assume without loss of generality that $X = \Lambda^{m,n}$ is a model supermanifold and that $0 < \epsilon < |\beta \circ \mathrm{ev}(f)(x)| < 1$ for each $x \in \Lambda^{m,n}$ and some fixed $\epsilon > 0$. The superfunction can be uniquely represented as $f = g + \phi$, where g is a G^∞ superfunction in even variables with the property $0 < \epsilon < |\beta \circ \mathrm{ev}(g)(x)| < 1$ for each $x \in \Lambda^{m,n}$, while ϕ is a nilpotent superfunction of nilpotency class $\leq 2^n$. Since in the following infinite sum each binomial expansion terminates after the first 2^n terms, one has

$$\begin{aligned}\sum_{j=0}^{\infty}(g+\phi)^j &= \sum_{j=0}^{\infty} g^j + \phi\left(\sum_{j=0}^{\infty} j g^{j-1}\right) + \cdots \\ &\quad + \phi^{2^n-1}\left(\sum_{j=0}^{\infty} j(j-1)\cdots(j-2^n+1) g^{j-2^n+1}\right),\end{aligned}$$

and the above series converges to the superfunction

$$\begin{aligned} g &= (1-g)^{-1} + \phi(1-g)^{-2} + 2\phi^2(1-g)^{-3} + \cdots \\ &\quad + (2^n-1)!\phi^{2^n-1}(1-g)^{-2^n+1}. \end{aligned}$$

All the coefficients in the latter polynomial in ϕ are G^∞ functions in even variables, and therefore g is a well-defined supersmooth superfunction on $\Lambda^{m,n}$. It is furthermore clear that $fg \equiv 1$. Finally, in order to get back from the local models to a global superfunction on X, notice that the uniqueness of the inverse element in an associative unital algebra assures the possibility to patch local pieces of f^{-1} together. □

To conclude this section, we provide an additional characterization of G^∞ functions of m even variables; notice that the sheaf $\hat{\mathcal{G}}_m^\infty$ of such functions coincides with the structure sheaf of the standard supermanifold $(\Lambda^{m,0}, \mathcal{G}_{m,0})$. If we define the sheaf $\mathcal{Q}$ on $\mathbb{R}^m$ as $\mathcal{Q} = \mathcal{G}_{m,0|\mathbb{R}^m}$, we have the inclusions $\mathcal{P} \subset \mathcal{Q} \subset \mathcal{C}^\infty_{\mathbb{K}^m} \otimes_{\mathbb{K}} \Lambda$, where $\mathcal{P}$ is the sheaf of Λ-valued Pringsheim regular functions on $\mathbb{K}^m$, and $\mathcal{C}^\infty_{\mathbb{K}^m}$ is the sheaf of $\mathbb{K}$-valued C^∞ functions on $\mathbb{K}^m$. The following fact is easily derived from Theorem 3.3 (cf. also [8]).

Proposition 4.4. *The sheaf $\hat{\mathcal{G}}_m^\infty$ is the image of $\mathcal{Q}$ under the Z-expansion.*

5. Cohomology of DeWitt supermanifolds

An R^∞ supermanifold X is called a *DeWitt supermanifold* if it admits an atlas whose charts are DeWitt open. For example, the model supermanifold is DeWitt.

Such supermanifolds over Λ_∞ formed historically the first ever class of supermanifolds over a non-trivial ground algebra [22] and since then have been studied rather extensively, particularly in the case where the ground algebra Λ is finite-dimensional [5,16–20,23,24,45]. In this section we show that the structure sheaf $\mathcal{A}$ of a DeWitt supermanifold M is acyclic, i.e. $H^k(M, \mathcal{A}) = 0$ for $k > 0$, where H^k denotes the kth sheaf cohomology functor. We shall follow quite closely [5] (cf. also [3]).

The following is easily derived from Corollary 3.5 in view of the fact that each R^∞ superfunction locally is a polynomial in odd variables whose coefficients are G^∞ functions.

Proposition 5.1. *R^∞ superfunctions preserve soul fibres. In full, if $f : \Lambda^{m,n} \to \Lambda^{p,q}$ is a superspace morphism between two model R^∞ supermanifolds, then for each $x, y \in \Lambda^{m,n}$ with $\beta^{m,n}(x) = \beta^{m,n}(y)$ one has $\beta^{p,q}(\tilde{f}(x)) = \beta^{p,q}(\tilde{f}(y))$.*

Let M be a DeWitt supermanifold with $\dim M = (m, n)$. Say that two points, x and y, are *equivalent*, $x \sim y$, if there exists a local coordinate chart $\phi : V \to \Lambda^{m,n}$ satisfying $\beta^{m,n}(\phi(x)) = \beta^{m,n}(\phi(y))$. Proposition 5.1, applied to the transition functions of M, implies that $\sim$ is a well-defined equivalence relation on M. Define by $M_{\mathrm{B}} = M/\sim$ the corresponding topological quotient space and by $\pi_M : M \to M_{\mathrm{B}}$ the projection mapping. It is evident that $\Lambda^{m,n}/\sim$ is canonically homeomorphic to the standard m-dimensional topological manifold $\mathbb{K}^m$ and therefore supports a natural structure of an m-dimensional C^∞ manifold. Observe also that for each $x \in M$, the fibre $\pi_M^{-1}(x)$ is entirely contained in a suitable coordinate chart on M, because M is DeWitt. (In fact, the projection mapping

$\pi_M : M \to M_B$ determines the DeWitt topology on M in the sense that a subset $U \subseteq M$ is DeWitt open if and only if $U = \pi_M^{-1}\pi_M(U)$ and $\pi_M(U)$ is open in M_B.) This implies that M_B supports a structure of a finite-dimensional C^∞ manifold. It is called the *body (manifold)* of M. The above construction is well-known (cf. [5]), and we have repeated it here only to convince the reader that it still works if an AMGO algebra is chosen as the ground algebra of the theory.

Lemma 5.2. *Every DeWitt R^∞-supermanifold admits a coarse R^∞ partition of unity.*

Proof. Let M be a DeWitt R^∞-supermanifold. Using the paracompactness of M_B and standard tools of uniform topology, one can choose (i) a compatible metric ρ on M_B, and (ii) an open cover, γ, of M_B such that the open balls, $\mathcal{O}_1(V)$, of radius 1 around all $V \in \gamma$ are compact, form a locally finite cover of M_B, and each cylinder, $\pi_M^{-1}(\mathcal{O}_1(V))$, is a coordinate neighbourhood for M.

Now choose for each $V \in \gamma$ a bell function $f_V : \mathbb{R}^m \to \mathbb{R}$ such that $\mathrm{supp}(f_V \circ \phi_V) \subseteq \mathcal{O}_1(V)$ and $f_V \circ \phi_V|_V \equiv 1$, where $\phi_V : V \to \mathbb{R}^m$ denotes the respective coordinate chart. (This can be done easily by forming in $\mathbb{R}^n$ the convolution product of the characteristic function of V with a standard m-dimensional bell function, appropriately normalized.) Every bell function is Pringsheim regular, and therefore f_V admits a Z-expansion f_V^* to $\Lambda^{m,0}$. The composition, φ_V, of f_V^* with the corresponding coordinate chart $\tilde{V} \to \Lambda^{m,n}$ and the projection $\Lambda^{m,n} \to \Lambda^{m,0}$ is an R^∞ superfunction on M having the properties supp $(\varphi_V) \subseteq \mathcal{O}_1(V)$ and $\beta(\varphi_V)|_V \equiv 1$. Since the cover by cylinders $\pi_M^{-1}(\mathcal{O}_1(V))$ is locally finite, the sum $\tau = \sum_{V\in\gamma}\varphi_V$ is a correctly defined R^∞ superfunction on M. Since γ is a cover of M_B, one concludes that for each $x \in M$, $\beta(\tau(x)) \geq 1$. (Indeed, the body of the image of a point under an R^∞ superfunction remains constant along each soul fibre.) According to Lemma 4.3, $1/\tau$ is a correctly defined R^∞ superfunction on M. Normalizing the φ_V's by multiplying by $1/\tau$ supplies the desired coarse partition of unity by R^∞ superfunctions. □

Corollary 5.3. *The direct image sheaf $(\pi_M)_*\mathcal{A}$ on M_B is acyclic, i.e., $H^k(M_B, (\pi_M)_*\mathcal{A}) = 0$ for $k > 0$.*

We also need to know the cohomology of the standard supermanifold $(\Lambda^{m,n}, \mathcal{G}_{m,n})$.

Lemma 5.4. *The structure sheaf of the standard (m,n)-dimensional supermanifold is acyclic, i.e., $H^k(\Lambda^{m,n}, \mathcal{G}_{m,n}) = 0$ for $k > 0$.*

Proof. By Proposition 4.4 we have

$$\mathcal{G}_{m,n} \simeq (\beta^{m,n})^{-1}\mathcal{F}, \qquad \text{where } \mathcal{F} = \mathcal{Q} \otimes \wedge\mathbb{R}^n \otimes \Lambda.$$

Here $\beta^{m,n} : \Lambda^{m,n} \to \mathbb{R}^m$ is the body map, and $\mathcal{Q}$ is a subsheaf of $\mathcal{C}^\infty_{\mathbb{R}^m}$ which contains the sheaf $\mathcal{P}$ of Pringsheim regular functions on $\mathbb{R}^m$ (cf. discussion at the end of Section 4). If we denote $Y = \mathfrak{N}^{m,n}$, where $\mathfrak{N} = \mathfrak{J}_\Lambda$ is the nilpotent ideal in Λ, and by $\sigma : \Lambda^{m,n} \to Y$

the projection (soul map), for all $n \geq 0$ by standard homological algebra we have an exact sequence

$$0 \to \bigoplus_{j+k=n} H^j(\mathbb{R}^m, \mathcal{F}) \otimes_{\mathbb{Z}} H^k(Y, \mathbb{Z}) \to H^n(\Lambda^{m,n}, \sigma^{-1}\mathcal{F})$$
$$\to \bigoplus_{j+k=n+1} \mathrm{Tor}_{\mathbb{Z}}[H^j(\mathbb{R}^m, \mathcal{F}), H^k(Y, \mathbb{Z})] \to 0.$$

As $\mathbb{Z}$ over Y is acyclic, and $H^0(Y, \mathbb{Z}) \simeq \mathbb{Z}$, we obtain from the previous sequence

$$H^k(\Lambda^{m,n}, \mathcal{G}_{m,n}) \simeq H^k(\mathbb{R}^m, \mathcal{F}).$$

Since the sheaf of rings $\mathcal{P}$ is fine (i.e. there exists partitions of unity made up by Pringsheim regular functions), the sheaf $\mathcal{F}$ of $\mathcal{P}$-modules is soft, hence acyclic. □

Proposition 5.5. *The structure sheaf $\mathcal{A}$ of a DeWitt supermanifold M is acyclic.*

Proof. Any $p \in M_{\mathrm{B}}$ has a system of neighbourhoods $\{U\}$ such that for all U, the supermanifold $((\pi_M)^{-1}(U), \mathcal{A}_{|(\pi_M)^{-1}(U)})$ is isomorphic to the standard supermanifold $(\Lambda^{m,n}, \mathcal{G}_{m,n})$. Then by Lemma 5.4 the sheaf $\mathcal{A}_{|(\pi_M)^{-1}(U)}$ is acyclic. By Leray's lemma [5,26] we have

$$H^k(M, \mathcal{A}) \simeq H^k(M_{\mathrm{B}}, (\pi_M)_*\mathcal{A}), \qquad k \geq 0.$$

By Lemma 5.3 we conclude. □

As in the case of a finite-dimensional ground algebra [2], Proposition 5.5 implies – via the so-called abstract de Rham theorem – that the (super) de Rham cohomology of M is isomorphic to the Čech cohomology $\check{H}^\bullet(M, B)$, which is in turn isomorphic to the ordinary de Rham cohomology of (the smooth manifold underlying) M tensored by B. Now, M is a locally trivial bundle on M_{B} with a vector fibre [5], so that M is contractible to M_{B}, and the latter cohomology is isomorphic to the de Rham cohomology of M_{B}. Let Ω^k_M be the sheaf of k-superforms on M, and denote by $H^\bullet_{SDR}(M)$ the cohomology of the differential complex $(\Omega^\bullet_M(M), d)$, where d is the exterior differential.

Corollary 5.6. $H^k_{SDR}(M) \simeq H^k_{DR}(M_{\mathrm{B}}) \otimes \Lambda, \quad k \geq 0.$

6. A correspondence between graded manifolds and DeWitt supermanifolds

The first aim of this section is to establish the following result, previously known for finite-dimensional ground algebras [5].

Theorem 6.1. *The isomorphism classes of smooth DeWitt supermanifolds over a fixed AMGO algebra and those of smooth graded manifolds are in a natural one-to-one correspondence with each other.*

Lemma 6.2. *Let M be an H^∞ DeWitt supermanifold, with structure sheaf $\mathcal{H}$ and body projection $\Phi : M \to M_B$. Then the pair $(M_B, \Phi_*\mathcal{H})$ is a graded manifold.*

Proof. An augmentation map $\sim: \Phi_*\mathcal{H} \to C^\infty_{M_B}$ is defined by setting $\tilde{f}(\Phi(p)) = \beta(f(p))$, where β is the body map. The local triviality of $(M_B, \Phi_*\mathcal{H})$ is ensured by the isomorphism $(\Phi_*\mathcal{H})_{|U} \simeq C^\infty_{M_B|U} \otimes \wedge(n)$ holding for small enough open sets $U \subset M_B$. All defining properties of graded manifolds are then satisfied. □

Lemma 6.3. *Let $(X, \mathcal{A})$ be a graded manifold. One can construct an H^∞ DeWitt supermanifold M, with structure sheaf $\mathcal{H}$ and body projection $\Phi : M \to M_B$, such that $X \simeq M_B$ and $\Phi_*\mathcal{H} \simeq \mathcal{A}$.*

Proof. It is known that every smooth manifold admits an analytic structure [53] (cf. also [11]). By realizing the sheaf $\mathcal{A}$ as an exterior bundle $\mathcal{A} = \wedge\mathcal{E}$ (here $\mathcal{E} = \mathcal{N}/\mathcal{N}^2$, where $\mathcal{N}$ is the nilpotent subsheaf of $\mathcal{A}$; cf. [10]), we may put on $(X, \mathcal{A})$ an analytic atlas. Then, if one considers the transition functions of this atlas, and, after expanding them in powers of the odd coordinates, takes the Z-expansions of the coefficients, these will converge. This yields H^∞ transition functions for M (for details see [5]). □

Proof of Theorem 6.1. In view of the two previous lemmas one has only to establish a bijection between the sets of isomorphism classes of H^∞ and R^∞ DeWitt supermanifolds. One easily checks that if $(M, \mathcal{H})$ is an H^∞ DeWitt supermanifold, then $(M, \mathcal{H} \otimes \Lambda)$ is an R^∞ DeWitt supermanifold. Conversely, if $(M, \mathcal{A})$ is an R^∞ DeWitt supermanifold, then $(M, \mathcal{A} \otimes_\Lambda \mathbb{K})$ (where $\mathbb{K}$ is a Λ-module via the body map) is an H^∞ DeWitt supermanifold. □

It is useful to notice that the proofs of Lemma 6.2 and of Theorem 6.1 yield slightly more than is contained in the statements of results alone. The following is yet another useful – and immediate – by-product of the same argument.

Proposition 6.4. *The correspondence $(M, \mathcal{G}) \mapsto (M_B, \mathcal{A})$ determines a covariant functor from the category of R^∞ DeWitt supermanifolds over a fixed AMGO algebra and superspace morphisms to the category of graded manifolds and their morphisms.*

In the remaining part of the section we will demonstrate that, contrary to what was claimed in [35], only in special situations can the functor of change of base be applied to graded manifolds in order to obtain DeWitt supermanifolds. There are some subtleties in this transition that were overlooked by the author of [35].

Both the functor of points and the functor of change of base come from algebraic geometry and theory of analytic spaces *à la Grothendieck* [21]. Here we present them in the form they assumed in supergeometry [14,35]. The need for the functor of points approach is explained by the fact that *superfunctions – and therefore morphisms between superspaces – are not uniquely determined by the collection of their values.* (A similar phenomenon occurs in algebraic geometry for the nilpotents of the structure sheaves of non-reduced schemes.)

Let Λ be an arbitrary AMGO algebra. Denote by SpecΛ a locally ringed superspace (i.e., a superspace over $\mathbb{K}$ as the ground algebra) whose underlying topological space is a singleton and the structure sheaf is constant, with Λ as the algebra of global sections. In particular, Spec$\wedge(q) = \mathbb{K}^{0,q}$ is a purely odd model supermanifold of dimension $(0, q)$, also sometimes denoted by pt_q.

Let now X be an arbitrary locally ringed superspace. Any superspace morphism Spec$\Lambda \to X$ is called a *Λ-point* of X. This terminology is justified by the fact that pt_0-points are in a natural one-to-one correspondence with the points of the underlying topological space of X. (Because the structure sheaf is one of local graded algebras, every stalk admits a unique unital algebra homomorphism to the basic field, and therefore a morphism $pt_q \to X$ is uniquely determined by the underlying mapping $\{*\} \to X$.) Denote the collection of all Λ-points of X by $pt_\Lambda(X)$. The correspondence $\Lambda \mapsto pt_\Lambda(X)$ from the category of all local graded-commutative algebras to the category of all sets and mappings, *Sets*, determines a contravariant functor. Also, if a ground algebra Λ is fixed, then the correspondence $X \mapsto pt_\Lambda(X)$ is functorial as well: every superspace morphism $f : X \to Y$ gives rise to a set-theoretic mapping $pt_\Lambda(f) : pt_\Lambda(X) \to pt_\Lambda(Y)$ via the formula $pt_\Lambda(f)(\kappa) = f \circ \kappa$ for each κ : Spec $\Lambda \to X$.

Let us try to compute Λ-points of some superspaces. We begin with the simplest nontrivial graded manifold, that of dimension (0, 1). It is easy to see that Λ-points of $\mathbb{K}^{0,1}$ are in a one-to-one correspondence with elements of $\Lambda^{0,1}$: indeed, each Λ-point is essentially a graded algebra homomorphism $\wedge(1) \to \Lambda$ and as such is uniquely determined by the image of a fixed odd generator $\xi \in \wedge(1)$ in Λ_1. This argument leads to the following result.

Lemma 6.5. *Homorphisms from $\Lambda(n)$ to a graded-commutative algebra Λ – and therefore Λ-points of the superspace pt_q – are in a natural one-to-one correspondence with elements of $\Lambda^{0,n}$.*

Next we show that even computing the set of Λ-points for the purely even smooth manifold $\mathbb{R}^{1,0} = \mathbb{R}$ turns out to be a subtler task than it might seem at first.

Lemma 6.6. *Let $\mathcal{O}_n$ denote the algebra of germs of infinitely smooth real functions on $\mathbb{R}^n$ at the origin. Let Λ be a local graded-commutative algebra with the property that every element of the radical is nilpotent. Homomorphisms from $\mathcal{O}_n$ to Λ are in a natural one-to-one correspondence with elements of $J_V^{n,0}$.*

Proof. Denote by x_i, $i = 1, \ldots, n$ the germs of coordinate functions at 0. Let α_i, $i = 1, \ldots, n$ be a collection of elements of J_Λ. Denote by N a natural number with the property $\alpha_i^N = 0$ for each $i = 1, \ldots, n$. If $P_N(x_1, \ldots, x_n)$ is the Taylor polynomial of degree N for f at zero, set $h_{\alpha_1,\ldots,\alpha_n}(f) = P_N(\alpha_1, \ldots, \alpha_n) \in \Lambda$. Clearly, $h_{\alpha_1,\ldots,\alpha_n}$ is a graded unital algebra homomorphism. On the other hand, if $h : \mathcal{O}_n \to \Lambda$ is an arbitrary graded unital algebra homomorphism, set $\alpha_i = h(x_i)$ for each $i = 1, \ldots, n$. It is not difficult to see that $h = h_{\alpha_1,\ldots,\alpha_n}$, which finishes the proof. □

The following is an immediate corollary of Lemmas 6.5 and 6.6.

Proposition 6.7. *Let Λ be an AMGO algebra having weakly pronilpotent radical. Let m, n be natural numbers. The Λ-points of $\mathbb{R}^{m,n}$ are in a natural one-to-one correspondence with the elements of the linear superspace $\Lambda^{m,n}$. Symbolically, $pt_\Lambda \mathbb{R}^{m,n} \cong \Lambda^{m,n}$.*

In particular, the above result holds for finite-dimensional Grassmann algebras, as well as for DeWitt's algebra Λ_∞ and for the algebra $\wedge(\infty)$. Surprisingly, this ceases to be true in the absence of weak pronilpotency of the radical, even for such a 'nice' algebra as Rogers' Banach–Grassmann algebra B_∞.

Example 6.8. The B_∞-points of the graded manifold $\mathbb{R}^{1,0}$ are in a natural one-to-one correspondence with those elements of the model supermanifold $B_\infty^{1,0}$ having nilpotent soul. In particular, the set $pt_{B_\infty}(\mathbb{R}^{m,n})$ does not support a natural structure of a DeWitt superdomain $B_\infty^{m,n}$, as it was claimed by Leites in [35] – in fact, $pt_{B_\infty}(\mathbb{R}^{m,n})$ is a proper everywhere dense subset of $B_\infty^{m,n}$.

To prove this, first observe that B_∞ admits a canonical continuous graded algebra monomorphism into DeWitt algebra Λ_∞, which we will denote by i. It gives rise to a dual morphism of superspaces, $\hat{i}$: Spec $\Lambda_\infty \to$ Spec B_∞, which is, from category-theoretic viewpoint, an epimorphism. By composing a B_∞-point of a superspace X with $\hat{i}$, one obtains a Λ_∞-point, and the emerging mapping $pt_{B_\infty} X \to pt_{\Lambda_\infty} X$ is into.

Let $x \in (B_\infty)_0$ be such that $\sigma(x)$ is nilpotent. In such a case, the Z-expansion converges at the point x for every infinitely smooth real function defined in the vicinity of $\beta(x)$, and the value of $Z(f)$ at x gives us the desired morphism from the stalk at $\beta(x)$ to Λ:

$$(f)_{\beta(x)} \mapsto (Z(f))(x).$$

This allows us to assign to $x \in (B_\infty)_0$ the morphism Spec $B_\infty \to \mathbb{R}^{0,1}$ of the form $* \mapsto \beta(x)$, $(f)_{\beta(x)} \mapsto (Z(f))(x)$.

Now let $\mu = (* \mapsto y, \varphi)$ be any superspace morphism, where $\varphi : \mathcal{C}_y \to (B_\infty)_0$ is an algebra morphism. According to the proof of Proposition 6.7, a homomorphism $i \circ \varphi$ from $\mathcal{C}_{\beta(x)}$ to Λ_∞ is uniquely determined by its value on the germ of the local coordinate function $z \mapsto z - y$, and one must have in fact $(i \circ \varphi)(f) = (Z(f))(x)$, where the Z-expansion is taken in Λ_∞. Consequently, the same must hold in B_∞. But as we have seen before, for a non-nilpotent x the value $Z(f)$ is undefined in B_∞: what happens is that the series $Z(f)$ converges in the larger algebra Λ_∞, and the value of the sum belongs to B_∞ for every f if and only if $\sigma(x)$ is nilpotent.

Now we get back to Arens–Michael algebras having weakly pronilpotent radical. Let $m \in \mathbb{N}$. If $f : \mathbb{R}^{(m,n)} \to \mathbb{R}^{(p,q)}$ is a superspace morphism, then the induced mapping

$$pt_\Lambda(f) : pt_\Lambda(\mathbb{R}^{(m,n)}) \equiv \Lambda^{m,n} \to \Lambda^{p,q} \equiv pt_\Lambda(\mathbb{R}^{(p,q)})$$

is a G^∞ mapping, i.e., a morphism of Rogers G^∞-superspaces over Λ. The mapping $pt_\Lambda(f)$ in itself need not determine a superspace morphism; this is only the case if in addition

$\Lambda^\perp = \{0\}$, cf. Proposition 4.2. However, one can easily get round this obstacle. Indeed, the projective tensor product $\Theta = \Lambda\hat{\otimes}_\pi \wedge(\infty)$ forms an AMGO algebra (Proposition 2.3), satisfying the desired property $\Lambda^\perp = \{0\}$ and in addition having weakly pronilpotent radical if Λ had this property. Consequently, the G^∞-mapping $pt_\Theta(f) : \Theta^{m,n} \to \Theta^{p,q}$ determines in a unique way a morphism between R^∞-supermanifolds (over Θ). Now it is easy to verify the following. Recall that Λ is contained in Θ as a topological graded subalgebra in a canonical way: $\Lambda \ni x \mapsto x \otimes 1_M \in \Lambda \otimes M \subset \Theta$.

Proposition 6.9. *Let Λ be an AMGO algebra having weakly pronilpotent radical. Superfunctions on a DeWitt open subset $\widetilde{U} \subseteq \Lambda^{p,q}$ are in a natural one-to-one correspondence with those G^∞ functions f on the corresponding DeWitt open subset $\bar{U} = \beta_\Theta^{-1}(U)$ taking values in Θ for which $f(U) \subseteq \Lambda$.*

Proof. If f is a G^∞ superfunction, then the Z-expansion of $f|_U$, taken in $\Theta^{p,q}$, is a G^∞ superfunction on $\bar{U}$, obviously having the property $Z_\Theta(f)(U) \subseteq \Lambda$. If f is a purely odd coordinate superfunction of the form $(x_1, \ldots, x_m, \xi_1, \ldots, \xi_n) \mapsto \xi_j$, then its extension to Θ is the corresponding coordinate function, and the desired property is obvious. Since each superfunction on U is a polynomial in odd variables with G^∞ superfunctions as coefficients, the extension rule is now uniquely determined by the property of being a Λ-algebra homomorphism. Its injectivity is practically evident. Concerning surjectivity, it suffices to check it for G^∞ superfunctions f over Θ only, but the restrictions of such superfunctions with the property $f(U) \subseteq \Lambda$ to $\widetilde{U}$ are genuine G^∞ superfunctions over Λ. This completes the proof. □

It follows that $pt_\Theta(f)$ determines a Λ-superspace morphism, which we shall denote by the same symbol $pt_\Lambda(f) : \Lambda^{m,n} \to \Lambda^{p,q}$. To compute the pullback of a superfunction g on a DeWitt open subset $\widetilde{U} \subseteq \Lambda^{p,q}$ under $pt_\Lambda(f)$, we first extend g to a Θ-valued superfunction, g', on $\bar{U}$, and then take as $pt_\Lambda(f)^\sharp(g)$ the perfunction on $pt_\Lambda(f)^{-1}(U)$ corresponding to the G^∞ superfunction $g' \circ pt_\Theta(f)$ on $pt_\Lambda(f)^{-1}(U)$.

Applying the above observation to the transition superfunctions of an atlas of a graded manifold, one can show that for every smooth Berezin–Leĭtes–Kostant supermanifold X, the set $pt_\Lambda(X)$ supports a natural structure of an R^∞-supermanifold over Λ. Moreover, this supermanifold is DeWitt. Furthermore, one can easily prove that the construction of the previous section is actually functorial at least when Λ is restricted to the class of AMGO algebras with weakly pronilpotent radical.

Theorem 6.10. *Let Λ be an AMGO algebra having weakly pronilpotent radical. The correspondence $pt_\Lambda(f)$ is a functor from the category of Berezin–Leĭtes–Kostant smooth supermanifolds (graded manifolds) to the category of R^∞ DeWitt supermanifolds over Λ.*

Remark 6.11. *For general AMGO algebras we can only claim that this correspondence is functorial in the real analytic case.*

The functor $pt_\Lambda(-)$ is called a *change of base functor.* The above facts were known in folklore, cf. [14,35], but at the intuitive level only, it seems. Such a link between various approaches to supermanifolds was first pointed out by Leĭtes [35] and (independently) Schwarz [51], and it remains largely unexplored to date. In a somewhat different form, this construction was also known to DeWitt [23].

It is still important to remember, however, that there are, in general, many more morphisms between two DeWitt supermanifolds than between the two corresponding graded manifolds, i.e., not every morphism between DeWitt supermanifolds $pt_\Lambda(X)$ and $pt_\Lambda(Y)$ is of the form $pt_\Lambda(f)$ for a suitably chosen morphism $f : X \to Y$. In fact, this phenomenon can be already observed for the model supermanifolds in dimension $(1, 0)$ or $(0, 1)$.

Finally, let us note that supermanifolds over a fixed AMGO algebra form a category with direct products, cf. [6].

7. Super Lie groups and deformations

A *supergroup* of finite dimension is a group object in the category of supermanifolds, i.e., a quadruple $(G, \mu, \nu, \varepsilon)$, where $\mu : G \times G \to G$ is a morphism of supermanifolds (the multiplication morphism), $\nu : G \to G$ is another morphism (the inversion morphism), as is $\varepsilon : pt_0 \to G$ (the unity morphism); those morphisms satisfy the usual group object axioms of associativity, existence of unity and existence of inverses, expressed by means of commutative diagrams (cf. [14]). To every supergroup one can associate in a natural way its Lie superalgebra, formed by all left-invariant vector superfields on G. (Left-invariance can no longer be expressed through left translations alone, and requires an elaborate trick in order to be defined, cf. [13,14,31].)

A direct application of the functor of change of base to G leads, modulo Theorem 6.10, to the following result.

Theorem 7.1. *Let Λ be an Arens–Michael algebra of Grassmann origin having pronilpotent radical. For every graded group G, the DeWitt supermanifold $pt_\Lambda(G)$ supports the natural structure of a DeWitt supergroup over Λ. The correspondence $G \mapsto pt_\Lambda(G)$ is a covariant functor from the category of graded groups and their morphisms to the category of DeWitt supergroups over Λ and supergroup morphisms.*

DeWitt [23,24] calls those supergroups contained in the image of the functor $pt_\Lambda(-)$ *conventional.* Not all supergroups are conventional; for examples we refer the reader to DeWitt's book [23] and the paper [24].

On the other hand, if G is a supergroup over an AMGO algebra, Λ, then the application of Proposition 6.4 immediately leads to the existence of a graded group structure on the body Lie group, G_B. Using the terminology of [25], we can state this result as follows.

Theorem 7.2. *Let Λ be an AMGO algebra. Each DeWitt supergroup over Λ is a deformation of a suitable graded group over* Spec Λ.

To understand the meaning of the word 'deformation' in the present context, notice that, as an $\mathbb{R}$-superspace, every DeWitt supergroup G is the direct product of its body graded manifold G_B and the superspace Spec Λ. If one denotes by $\varpi_1 : G \cong G_B \times \text{Spec}\,\Lambda \to G_B$ the first coordinate projection, and by μ and μ_B the multiplication morphisms on G and G_B, respectively, then one has $\varpi_1 \circ \mu \circ (\varpi_1 \times \varpi_1) = \mu_B$. In other words, the multiplication morphism on G preserves the fibres of the superfibre bundle $G_B \times \text{Spec}\,\Lambda \to G_B$, and in this sense it is a deformation of the natural multiplication on the direct product of supergroups G_B and Spec Λ. (The latter object, though not a supergroup in a strict sense, is a group object in the category of superspaces with respect to a natural abelian supergroup structure.)

The validity of Theorem 7.2 was conjectured by the second named author (VP) in [44].

An interesting open question is: can every graded group be deformed into an 'unconventional' DeWitt supergroup? In other words, given a graded group G, does there exist a DeWitt supergroup, $\hat{G}$, over the ground algebra Λ_∞, such that the body graded group, $\hat{G}_B$, is isomorphic to G, and yet $\hat{G}$ is unconventional, i.e., $\hat{G} \not\cong pt_{\Lambda_\infty}(G)$?

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